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THE TORSION OF A NON-HOMOGENEOUS STRATUM WITH A HYPERBOLIC VARIATION OF SHEAR MODULUS

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Abstract—An exact formulation of the governing dual integral equations for the torsion of a non-homogeneous stratum due to a rigid circular body at its free surface is presented. The stratum varies in shear modulus according to the hyperbolic variation in a contemporary work[1]. It is shown that the unknown static stress distribution under the rigid body is governed by modified Bessel function of the first kind. By comparing the governing functions in the dual integral equations for five cases of elastic media: homogeneous half-space, and stratum, linearly non-homogeneous half-space and stratum and, finally, the present non-homogeneous stratum with hyperbolic variation, it is established that the surface shear modulus is the dominant parameter in the assessment of the stress and displacement fields in a non-homogeneous stratum where lateral variation of elastic properties is negligible.

1. INTRODUCTION

In a contemporary work[1], the author has considered the problem of static stress distribution under a rigid rectangular body resting on the free surface of an infinitely wide elastic stratum in which the shear modulus increases only in the depth direction, z, according to the hyperbolic variation:

$$G(z) = \frac{G_0 h}{h - z}. \tag{1}$$

The major conclusion of that work is to establish the dominance of surface modulus as a good first approximation to the solution of the unknown stress distribution under the body. The result, however, contains two fundamental unknown elastic properties: modulus of elasticity and Poisson's ratio. As it is well-known, the torsional case provides a useful avenue for separating the two unknowns since it is independent of Poisson's ratio. Therefore, if the conclusion of the work[1] is still valid for the torsional case as, indeed, it has been suggested in a previous work[2], then we can use the result of torsional case to evaluate shear modulus from which Poisson's ratio can be determined using results of the translational cases.

The present investigation is, therefore, concerned with the problem of the torsion of a non-homogeneous stratum with a shear modulus variation given by equation (1) due to the application of a static torque to a rigid circular body resting on its free surface. The stratum clearly merges into a rigid bed at depth h.

It is shown that the governing function in the dual integral equations is very close to corresponding functions for three other cases: a homogeneous elastic stratum of depth h, a linearly non-homogeneous stratum of depth h in which the shear modulus at the base is twice the surface shear modulus and, finally, a linearly non-homogeneous half-space with the same rate of increase of modulus as in the stratum. The last two cases have been chosen because their rate of

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increase of modulus is the same (G_0/h) as the rate at the surface of the hyperbolic variation in equation (1). The dominance of surface shear modulus is then investigated by comparing all the four cases with the governing function of a homogeneous half-space whose modulus is the same as the surface modulus of the non-homogeneous media.

2. GOVERNING DIFFERENTIAL EQUATIONS

We consider the stratum as having an arbitrary variation $G(z)$ which will be specified later. The elastic equations in cylindrical polar coordinates, for a stratum at rest, are

$$2G\frac{\partial \omega_z}{\partial r} - 2G\frac{\partial \omega_r}{\partial z} + \gamma_{\theta z}\frac{dG}{dz} = 0 \tag{2}$$

where the components of rotation and of shear strain are related to the only non-vanishing component of displacement v in the θ-direction by

$$\begin{aligned} 2\omega_r &= -\frac{\partial v}{\partial z} \\ 2\omega_z &= \frac{1}{r}\frac{\partial}{\partial r}(vr) \\ \gamma_{\theta z} &= \frac{\partial v}{\partial z}\,. \end{aligned} \tag{3}$$

Substituting these in equation (2), we can show that

$$\frac{\partial^2 v}{\partial z^2} + \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(vr)\right] + \frac{1}{y}\frac{\partial v}{\partial z} = 0 \tag{4}$$

where we have introduced

$$y(z) = \frac{G(z)}{G'(z)}. \tag{5}$$

Assuming now the hyperbolic variation of equation (1) for $G(z)$, we find from equation (5) that

$$y(z) = h - z. \tag{6}$$

Using now $y(z)$ as a subsidiary independent variable and noting that

$$\frac{\partial v}{\partial z} = -\frac{\partial v}{\partial y} \tag{7}$$

we transform equation (4) into

$$\frac{\partial^2 v}{\partial y^2} + \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(vr)\right] - \frac{1}{y}\frac{\partial v}{\partial y} = 0. \tag{8}$$

Introducing now the dependent variable $\bar{v}(p, y)$ of Hankel transform of order unity defined by

$$\bar{v}(p, y) = \int_0^\infty v(r, y) r J_1(pr)\, \mathrm{d}r \tag{9}$$

we reduce equation (8) to

$$\frac{\mathrm{d}^2\bar{v}}{\mathrm{d}y^2} - \frac{1}{y}\frac{\mathrm{d}\bar{v}}{\mathrm{d}y} - p^2\bar{v} = 0 \tag{10}$$

which is the governing differential equation in terms of the displacement transform.

3. GENERAL SOLUTION OF THE GOVERNING EQUATION AND EXPRESSION FOR STRESS TRANSFORM

Introducing the subsidiary variables V and Y defined through

$$\left.\begin{aligned} \bar{v}(p, y) &= \mathrm{e}^{-py} V(Y) \\ Y &= 2py \end{aligned}\right\} \tag{11}$$

we can show that equation (10) reduces to Kummer's equation

$$Y\frac{\mathrm{d}^2 V}{\mathrm{d}Y^2} - (1 + Y)\frac{\mathrm{d}V}{\mathrm{d}Y} + \frac{1}{2}V = 0. \tag{12}$$

However, the solutions of equation (12) in terms of the confluent hypergeometric functions of the first and second kinds $\Phi(a, c; x)$ and $\Psi(a, c; x)$ present a little difficulty because the parameter c becomes a negative integer:

$$c = -1. \tag{13}$$

Using the notation of Erdélyi *et al.*[3], the appropriate solution of the Φ-function is

$$y_2 = x^{1-c}\Phi(a - c + 1, 2 - c; x) \tag{14}$$

in which, for our present case

$$\begin{aligned} c &= -1 \\ a &= -\frac{1}{2} \\ x &= 2py. \end{aligned} \tag{15}$$

Similarly for the Ψ-function, we require the transformation in ([3], p. 257, equation 6)

$$\Psi(a, c; x) = x^{1-c}\Psi(a - c + 1, 2 - c; x). \tag{16}$$

We find, using the above, that the general solution of equation (12) is

$$V = Y^2[A\Psi(1\tfrac{1}{2}, 3; Y) + B\Phi(1\tfrac{1}{2}, 3; Y)] \tag{17}$$

which gives

$$\bar{v}(p, y) = 4p^2y^2 e^{-py}[A\Psi(1\tfrac{1}{2}, 3; 2py) + B\Phi(1\tfrac{1}{2}, 3; 2py)]. \tag{18}$$

It is easy to recognise now that the Ψ and Φ functions which occur in equation (18) are related to modified Bessel function of the first and second kinds by virtue of the known relations ([3], p. 265, equations 10 and 12).

$$\left.\begin{aligned} I_\nu(x) &= \frac{1}{\Gamma(v+1)}(\tfrac{1}{2}x)^\nu\, e^{-x}\,\Phi[(\tfrac{1}{2}+\nu),(1+2\nu);2x] \\ K_\nu(x) &= \pi^{1/2} e^{-x}(2x)^\nu \Psi[(\tfrac{1}{2}+\nu),(1+2\nu);2x] \end{aligned}\right\} \tag{19}$$

so that, on setting

$$\nu = 1 \tag{20}$$

we find

$$\bar{v}(p, y) = 2py\left[A\frac{K_1(py)}{\sqrt{\pi}} + 4BI_1(py)\right]. \tag{21}$$

For the transform of the shear stress given by

$$\begin{aligned} \bar{\tau}_{z\theta} &= G(z)\frac{\mathrm{d}\bar{v}}{\mathrm{d}z} \\ &= -\frac{G_0 h}{y}\frac{\mathrm{d}\bar{v}}{\mathrm{d}y} \end{aligned} \tag{22}$$

we require the differential properties of the function $I_1(x)$ and $K_1(x)$ which are known from the recurrence relations:

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}[xI_1(x)] &= xI_0(x) \\ \frac{\mathrm{d}}{\mathrm{d}x}[xK_1(x)] &= -xK_0(x) \end{aligned}\right\}. \tag{23}$$

Using these results in equations (21) and (22) leads to:

$$\bar{\tau}_{z\theta} = 2p^2 G_0 h\left[\frac{A}{\sqrt{\pi}}K_0(py) - 4BI_0(py)\right]. \tag{24}$$

4. EXPRESSIONS FOR THE ARBITRARY FUNCTIONS FROM BOUNDARY STRESSES AND DISPLACEMENTS

The unknown shear stress distribution under the rigid circular body and the known zero shear stress outside it will be represented by the discontinuous stress $\tau(r)$ which is valid throughout the free surface. At the base of the stratum, the particles can only be regarded as fixed so that the displacement component v vanishes. Thus, the two equations for evaluating A and B are

$$\bar{\tau}_{z\theta}/_{z=0} = \overline{\tau(r)}\,(r \geq 0) \tag{25}$$

$$\bar{v}/_{z=h} = 0\,(r \geq 0). \tag{26}$$

Noting that $z = 0$ at the surface corresponds to $y = h$ and $z = h$ at the base corresponds to $y = 0$, we can express equations (25) and (26) after using the general expressions for $\bar{\tau}_{z\theta}$ and $\bar{v}$ in equations (24) and (21) respectively as:

$$\frac{A}{\sqrt{\pi}} K_0(ph) - 4BI_0(ph) = \frac{\overline{\tau(r)}}{2p^2G_0h} \tag{27}$$

$$\lim_{y\to 0} \left\{ \frac{A}{\sqrt{\pi}} [yK_1(py)] + 4B[yI_1(py)] \right\} = 0. \tag{28}$$

The behaviour of modified Bessel functions for small values of the argument are known, for example, ([3], p. 5, equation 12 and p. 9, equation 37):

$$I_\nu(z) = \sum_{m=0}^{\infty} (\tfrac{1}{2}z)^{2m+\nu}/[m!\Gamma(m+\nu+1)] \tag{29}$$

$$\begin{aligned} K_n(z) &= (-1)^{n+1} I_n(z) \log(\tfrac{1}{2}z) + \tfrac{1}{2} \sum_{m=0}^{n-1} (-1)^m (\tfrac{1}{2}z)^{2m-n} \frac{(n-m-1)!}{m!} \\ &\quad + \tfrac{1}{2}(-1)^n \sum_{m=0}^{\infty} (\tfrac{1}{2}z)^{n+2m} \frac{[\psi(n+m+1) + \psi(m+1)]}{m!(n+m)!} \end{aligned} \tag{30}$$

where, in equation (30), ψ is the logarithmic derivative of the gamma function.

These results lead to

$$\lim_{z\to 0} I_1(z) \sim \frac{z}{2} + 0(z^3) \tag{31}$$

$$\lim_{z\to 0} K_1(z) \sim \frac{1}{z} + 0(z) \tag{32}$$

where, in equation (32), we have used the elementary result

$$\lim_{z\to 0} z \log(z) = 0. \tag{33}$$

We can now see from equation (28) that

$$A = 0 \tag{34}$$

so that, from equation (27),

$$B = -\frac{\overline{\tau(r)}}{8p^2G_0hI_0(ph)}. \tag{35}$$

Thus, throughout the stratum

$$\bar{v}(p, y) \equiv 8pyBI_1(py)$$
$$\equiv -\frac{\overline{\tau(r)}yI_1(py)}{G_0phI_0(ph)} \tag{36}$$

and

$$\bar{\tau}_{z\theta}(p, y) \equiv -8p^2G_0hBI_0(py)$$
$$\equiv \overline{\tau(r)}\frac{I_0(py)}{I_0(ph)}. \tag{37}$$

As a simple check, we find on setting $y = h$ (or $z = 0$) at the surface:

$$\overline{\tau_{z\theta}} \equiv \overline{\tau(r)} \tag{38}$$

and at the base where $y = 0$ (or $z = h$)

$$\bar{v} \equiv 0 \tag{39}$$

where we have made use of equation (31) in equation (36).

We notice that both equations (38) and (39) agree with the boundary conditions in equations (25) and (26).

5. GOVERNING DUAL INTEGRAL EQUATIONS AND COMPARISON WITH OTHER ELASTIC MEDIA

The exact mixed boundary conditions at the free surface are:

$$\left.\begin{aligned} v(r, 0) &= r\theta_0\,(0 \leq r < R) \\ \tau_{z\theta}(r, 0) &= 0\,(r > R) \end{aligned}\right\} \tag{40}$$

where θ_0 is the constant angle of twist due to the applied static torque on the circular body.

From equation (36), we find, by setting $z = 0$ $(y = h)$ at the surface that

$$\bar{v}(p, h) \equiv -\frac{\overline{\tau(r)}I_1(ph)}{G_0pI_0(ph)} \tag{41}$$

Using the Hankel inversion theorem on equation (41) to recover $v(r, 0)$ and, similarly on $\overline{\tau_{z\theta}}(p, h)$ to recover $\tau(r)$, we find that the governing dual integral equations are

$$\left.\begin{aligned} \int_0^\infty \frac{I_1(ph)}{I_0(ph)}\overline{\tau(r)}J_1(pr)\,\mathrm{d}p &= -G_0r\theta_0\,.\,(0 \leq r < R). \\ \int_0^\infty \overline{\tau(r)}pJ_1(pr)\,\mathrm{d}p &= 0\,(r > R) \end{aligned}\right\} \tag{42}$$

We readily recover the result for a homogeneous half-space as h tends to infinity when use has been made of the known asymptotic results, ([4], p. 202 equations 1 and 2).

$$\lim_{z\to\infty} I_0(z), I_1(z) \sim (2\pi z)^{-1/2}\, e^z \tag{43}$$

leading to

$$\begin{aligned} \int_0^\infty \overline{\tau(r)}\, J_1(pr)\, dp &= -G_0 r\theta_0 (0 \le r < R) \\ \int_0^\infty \overline{\tau(r)} p J_1(pr)\, dp &= 0\, (r > R) \end{aligned} \tag{44}$$

We now compare the governing dual integral equations for five cases in order to assess the extent of the dominance of surface shear modulus in the unknown stress distribution under the rigid body. The cases are for the following elastic media:

(1) Homogeneous half-space.
(2) Homogeneous stratum of depth, h.
(3) Linearly non-homogeneous half-space.
(4) Linearly non-homogeneous stratum of depth h and with shear modulus at the base being twice at the surface.
(5) Non-homogeneous stratum with hyperbolic variation having a surface rate (G_0/h) as the constant rate of shear modulus increase in cases (3) and (4).

If we write the governing dual integral equations in these five cases as

$$\left.\begin{aligned} \int_0^\infty \zeta_i(ph)\, \overline{\tau(r)}\, J_1(pr)\, dp &= -G_0 r\theta_0 (0 \le r < R) \\ \int_0^\infty \overline{\tau(r)} p J_1(pr)\, dp &= 0\, (r > R) \end{aligned}\right\} \tag{45}$$

we find respectively from the following works that

$$\zeta_1(ph) = 1 \quad \text{([5], p. 38, equations (28))}$$

$$\zeta_2(ph) = \tanh(ph) \quad \text{([6], p. 372, equations (10a))}$$

$$\zeta_3(ph) = \frac{K_0(ph)}{K_1(ph)} \quad \text{([2], p. 241, equations (38))}$$

$$\zeta_4(ph) = \frac{I_0(2ph)K_0(ph) - I_0(ph)K_0(2ph)}{I_0(2ph)K_1(ph) - I_1(ph)K_0(2ph)} \quad \text{([2], p. 241, equation (35))}$$

$$\zeta_5(ph) = \frac{I_1(ph)}{I_0(ph)}. \quad \text{(equations (42) of present work).}$$

These results have been compared in Fig. 1, throughout the range of the integrating parameter by introducing the auxiliary variable:

$$\eta = ph \tag{46}$$

Tables of functions in (4) and (7) have been used for the Bessel functions in the range $0 < \eta \leq 10$ and the asymptotic expansions in the range $\eta > 10$.

6. DISCUSSION AND CONCLUSIONS

It is convenient to split our discussion of the results based on Fig. 1 into two main aspects: the effect of the shear modulus as a dominant parameter and the relative effect of the other two factors—stratum depth and rate of non-homogeneity.

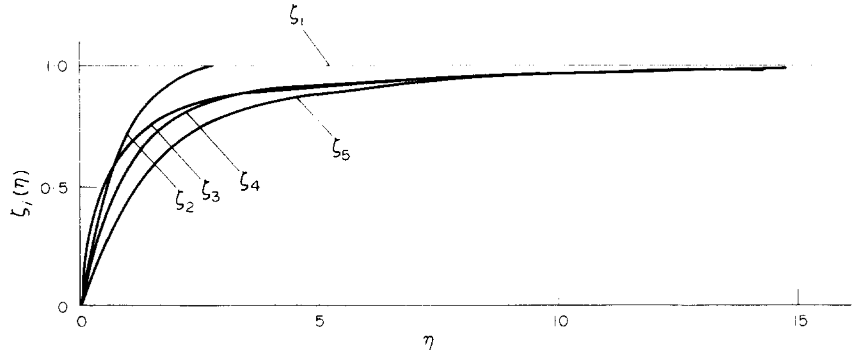

Fig. 1. Comparison of the governing functions for the five cases of elastic media.

Figure 1 shows that all the governing functions approach the value, unity, of the homogeneous half-space for large values of η, the difference in the range $\eta > 10$ is less than 5 per cent in all the cases. As expected, the stiffest medium given by ζ_5 is smallest in magnitude so that the corresponding unknown stress distribution will be the largest thus providing the greatest resisting torque when integrated over the circular area. The four functions ζ_2 to ζ_5 being all tied together both at the origin and at infinity show the secondary nature of the effect of stratum depth and non-homogeneity when compared with ζ_1 the homogeneous half-space.

In order to appreciate fully the dominance of the surface shear modulus, we remark here that Fig. 1 only demonstrates that the surface modulus effect is only a good first approximation. Indeed, to show that we can estimate the surface modulus of *any* non-homogeneous soil in which lateral variation is negligible from the theory of the homogeneous half-space, we observe that the whole integrand

$$I_i \equiv \zeta_i(ph)\,\overline{\tau(r)}\,J_1(pr),\; i = 1, 2, \ldots 5. \tag{47}$$

in equation (45) starts from the origin when use has been made of the power series of the Bessel functions and we record that the expression for $\overline{\tau(r)}$ in the case of the homogeneous half-space is given by:

$$\overline{\tau(r)} = -2\sqrt{\frac{2}{\pi}}\,G_0\theta_0R^{1\frac{1}{2}}p^{-1/2}J_{1\frac{1}{2}}(pR) \tag{48}$$

([5], p. 38, equation 28b).

Thus, the variation of the integrand over the whole range of the integrating parameter p is such that all the curves are tied together at the origin and infinity not excluding the homogeneous half-space unlike in Fig. 1.

We have shown in the previous work [2] that when the body on a non-homogeneous stratum is now subjected to harmonic oscillations even the secondary effect due to non-homogeneity and stratum depth is counteracted by the opposing effect due to the apparent increase in inertia of the body.

Our main conclusion, therefore, is that we can still estimate to a good degree of accuracy the surface shear modulus of a non-homogeneous soil from the simple expression:

$$G_0 = \frac{3\tilde{J}\rho R^2\Omega^2}{16} \tag{49}$$

of equation (61) in [2] where Ω is the resonant frequency, ρ is the soil density and $\tilde{J}$ is the non-dimensional polar inertia of the body.

REFERENCES

1. A. O. Awojobi, On the hyperbolic variation of elastic modulus in a non-homogeneous stratum. *Int. J. Solids Structures* (submitted 1974).
2. A. O. Awojobi, Torsional vibration of a rigid circular body on a non-homogeneous elastic stratum. *Quart. J. Mech. and Appl. Math.* **26**, 235, (1973).
3. A. Erdélyi, W. Magnus, F. Oberhenttinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. 2. McGraw-Hill, New York (1954).
4. G. N. Watson, *A Treatise on the Theory of Bessel Functions.* Cambridge University Press (1962).
5. A. O. Awojobi and P. Grootenhuis, Vibration of rigid bodies on semi-infinite elastic media. *Proc. Roy. Soc.* **A287**, 27 (1965).
6. A. O. Awojobi, Torsional vibration of a rigid circular body on an infinite elastic stratum. *Int. J. Solids Structures* **5**, 369 (1969).
7. G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers.* McGraw-Hill, New York (1961).